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# Cross-waves. Part 2. Experiments

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In this paper experiments in which cross-waves were generated in front of a wavemaker at one end of a long channel are described. The primary field produced was a progressive wave train, but, at certain frequencies, a standing wave developed in front of the wavemaker. This wave, whose crests were at right angles to the wavemaker and which had frequency half that of the wavemaker, is known as a cross-wave.

An instability mechanism for the formation of cross-waves in a long channel has been presented in Part 1 (Mahony 1972). In the present paper we describe some measurements concerning the formation of the waves. The initial growth rate of the waves has been measured and the curves of neutral stability determined for two of the possible modes. The results are in good agreement with Mahony's theory. After the early stages in the development of the waves, there is an increase in their growth rate. This new rate was found to be about twice the initial growth rate. Also reported are some measurements of the amplitude of the cross-wave field along the channel.

## 1. Introduction

Cross-waves are standing waves whose crests are at right angles to a wavemaker; they oscillate at half the frequency of the wavemaker. A discussion of the formation of these waves in a closed tank was given recently by Garrett (1970) and his results are in fairly good agreement with the experimental data. However, Mahony (1972) has shown that Garrett's mechanism for the formation of cross-waves is not applicable when the channel is very long and the primary field generated by the wavemaker is a progressive wave train. Accordingly Mahony proposed a new mechanism to account for the formation of cross-waves in a long channel. None of the existing experimental investigations provided adequate information to check the predictions arising from his analysis and the purpose of the present work is to check some of those predictions.

A survey of earlier experimental observations on cross-waves was included in § 1 of Garrett's paper, and in addition reference deserves to be made to a report by McGoldrick (1968) in which measurements on cross-waves of short wavelength are described. In particular, McGoldrick determined neutral-stability curves for the cross-wave modes $n = 24$ and $n = 25$, as explained below; his results are not closely comparable with the present ones, but appear to be in general qualitative agreement.

Some photographs of the phenomenon under discussion are shown in figure 1 (plate 1). These photographs, taken looking along the channel towards the wavemaker, clearly show the outline of the cross-wave at the wavemaker. The sides of the channel, whose width is 30·6 cm, are at the edges of the photographs. The mode numbers marked on the figure are the number of half wavelengths in the span of the channel; also indicated are the wavemaker periods at which the cross-waves were most easily excited. The decay of the cross-wave field along the channel can be seen in the photographs, especially for the first three modes. It is usual for the amplitudes of the cross-waves to be much larger than that of the progressive waves and this feature is shown in the photographs; in the case of the higher modes the progressive waves are hardly discernible. (It was not possible to drive the wavemaker at large enough amplitudes to generate the first ($n = 1$) mode spontaneously. However, with a careful choice of the frequency and with the wavemaker working at maximum amplitude, an artificially produced cross-wave was nearly maintained and decayed only at a very slow rate. For the sake of completeness this wave was used to illustrate the first mode in figure 1.)

Since the theory presented in Part 1 neglects surface tension, which on the scale of the experiments has an appreciable effect on the dispersion relation, we shall use the empirical value of the angular frequency $\Omega_n$ at which the relevant cross-wave is most easily excited as a basis for frequency scalings. It is assumed that the initial growth rate of the cross-waves is given by the difference between the theoretical growth rate, indicated by Mahony's theory, and a damping rate $L_n$. This damping rate is to be determined empirically. Then, for a plane wavemaker pivoting about a line on the bed of the channel and oscillating sinusoidally with an amplitude of $\theta$ radians, it follows from Part 1 (equation (14)) that the initial growth rate $\sigma_n$ of the $n$th mode is given by

$$\sigma_n = -L_n + \Omega_n \left\{ C_n^2 \theta^4 - \frac{1}{16}\left(\frac{\Omega_n^2 - \omega^2}{\omega^2}\right)^2 \right\}^{\frac{1}{2}}. \tag{1}$$

In this expression $\omega$ is the angular frequency of the wavemaker and $C_n$ is a constant given by

$$\left.\begin{aligned} C_n &= \tfrac{1}{64}\gamma_n^{-1}\{2\beta_n - \beta_n^2(\tfrac{3}{8} + 2n^2\alpha_n^2)\}^2, \\ \gamma_n &= \tfrac{1}{2}\{(n\alpha_n)^{-1}\tanh(n\alpha_n\beta_n) + \beta_n \operatorname{sech}^2(n\alpha_n\beta_n)\}, \\ \beta_n &= d\Omega_n^2/g, \quad \alpha_n = \pi g/(b\Omega_n^2), \end{aligned}\right\} \tag{2}$$

where $d$ is the depth of the channel, $b$ its breadth and $g$ the acceleration of gravity. Note that $C_n$ depends upon the size of the channel.

According to (1), the equation

$$L_n = \Omega_n C_n \Theta_n^2 \tag{3}$$

determines the smallest wavemaker amplitude $\Theta_n$ at which a cross-wave can be sustained. Moreover, it follows from (1) that the relationship expressing the margin of stability ($\sigma = 0$) is given, to a good approximation, in terms of the periods $T_n = 2\pi/\Omega_n$, $\tau = 2\pi/\omega$ by

$$(\tau - T_n)^2 = 4T_n^2 C_n^2(\theta^4 - \Theta_n^4). \tag{4}$$

In addition to checking these predictions from Mahony's theory, a brief investigation has also been made into some other features of the phenomenon. The analysis of Part 1 indicates that, far from the wavemaker, the initial amplitude of the cross-wave field decays as $\exp(-A\theta x)$, where $x$ is the distance from the wavemaker. Mahony suggests that this variation of the initial (growing) disturbance should be reflected, at least partially, in the established cross-wave field.

The above results relate only to the initial stages of development as, once the cross-waves have grown to a moderate size, a different mechanism for their growth comes into play. This mechanism, the explanation of which is outlined in Garrett's (1970) paper, entails the second-order pressures developed in a standing wave. These pressures, which oscillate at twice the frequency of the standing wave and are independent of the depth, are synchronized with the frequency of the wavemaker so that, when the phases are suitably disposed, a transfer of energy may take place from the wavemaker into the cross-waves. We have investigated the complete development process of the cross-waves in order to determine the importance of this second mechanism. The results of these measurements are discussed in § 3.

## 2. Experimental apparatus and procedure

A rectangular channel of width 30·6 cm and length 2·7 cm, containing water to a depth of about 16 cm, was used for the experiments. A plane flap, hinged at the channel bed, generated a progressive wave train at one end of the channel. A sloping beach was fitted at the other end to absorb the progressive waves, although viscous effects had dissipated a good proportion of the energy by the time the waves reached the beach. An adhesive cotton bandage was attached to the wavemaker and to the sides of the channel at the waterline. The bandage was water absorbant and helped reduce irregularities of the progressive wave train arising from uneven wetting of these surfaces. The beach was also covered with the bandage.

An electromagnetic vibrator (Pye–Ling model V50), powered by the amplified signal from a low frequency oscillator, was used to drive the wavemaker flap. Because the range of frequencies over which a particular cross-wave can be generated is very narrow, the frequency of the wavemaker had to be held constant to within close limits in order to obtain meaningful results. Thus, for the second mode, the cross-wave would be excited only if the short-term frequency stability of the wavemaker were better than 1 %, and in order to map out the curve of marginal stability ($\sigma_n = 0$), it was decided that the frequency drift of the wavemaker should not exceed about one part in one thousand during the time required to complete a measurement. The oscillator frequency was therefore monitored with an electronic counter, and on occasions when the frequency changed by more than 0·05 % during an experiment, the measurements were disregarded.

The amplitudes of the cross-waves were measured in one of two ways, depending upon the size of the waves. During the initial period of development, when their amplitudes were still small, a proximity meter (Wayne Kerr B731B) was used. The transducer was mounted above the water, just above the crests of the progressive waves, and in a position across the channel corresponding to an antinode of the cross-wave. This instrument converts the capacitance between the transducer and a plane surface into a voltage which is directly proportional to the distance between the two surfaces. In order to avoid large errors when using the instrument to determine the proximity of curved surfaces, the size of the transducer must be chosen carefully. For the present measurements a transducer was selected with an effective diameter of 11·3 mm, which ensured that, under the worst conditions, the position of a wave crest was known to within 2 %. The output from the proximity meter is linearly related to the distance between the two surfaces as long as the separation is less than the diameter of the transducer. This restricts the size of the waves that can be measured, but by looking only at the crests of the progressive waves the development of the cross-waves could easily be followed up to amplitudes of about 2 mm. Because the frequency of the cross-waves is exactly half that of the progressive waves, successive crests of the progressive wave train are separated by exactly half a cross-wave cycle. Thus the measurement of the heights of successive crests of the progressive waves provides a very convenient method of determining the cross-wave amplitude.

Larger cross-wave amplitudes were measured by a different method. A pair of vertical wires, extending from above the water surface to near the bed of the channel, were mounted on a carriage. The electrical resistance between each wire and an electrode in the water varied with the depth of water at the wire. These two wires were connected in opposite arms of an a.c. resistance bridge and, for large enough lengths of submerged wire, it was found that the a.c. signal from the bridge was proportional to the difference between the water levels at the two wires. Thus, when the two wires were placed at equal distances from the wavemaker the passage of the progressive waves produced no effect, and any voltage across the bridge arose from the presence of cross-waves. In making these measurements care must be taken that the menisci do not have a large influence on the readings.

Measurements of the damping rates of cross-waves were made by both the methods described above. A cross-wave was allowed to develop and the drive to the wavemaker was then turned off. After the progressive wave train had passed onto the beach a vertical baffle was placed in the channel in order to trap the cross-wave within a rectangular box and thereby prevent it from radiating to the beach. When the cross-wave had become uniform along this artificial box the measurement of its amplitude began. These measurements are further discussed in § 3.

## 3. Results

### 3.1. *Margin of stability*

The curves of neutral stability, delimiting the wavemaker amplitudes above which cross-waves are generated, were determined for the second and third modes.

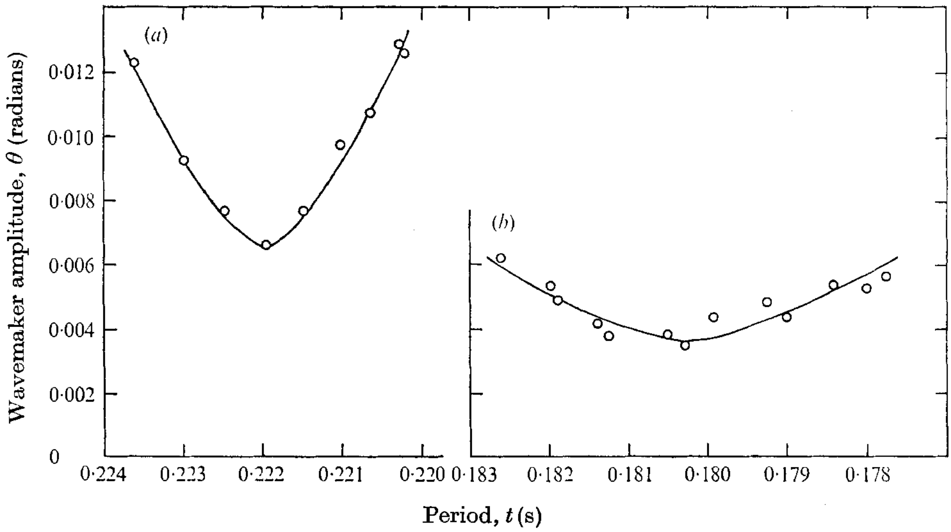


FIGURE 2. The margin of stability for two of the cross-wave modes. (*a*) The mode $n = 2$; the water depth for this experiment was 16·4 cm. (*b*) The mode $n = 3$; water depth $d = 16{\cdot}1$ cm.

Attention was concentrated on these modes because the wavemaker amplitudes needed to excite the first mode were beyond the range of the equipment; at full amplitude the amplifier delivered about 30 watts r.m.s. power.

A crucial factor was found to be the degree of contamination of the water surface. This has a profound effect on the progressive waves and can have an important influence on the measurements, especially those made on the higher modes. Thus, for example, while operating at the frequency $\Omega_3$ and starting with a 'clean' surface, it was found that the progressive waves showed no perceptible changes over a period of about 12 min, but from then on the amplitude of these waves at a given position began to decrease, so that after 30 min their amplitude had fallen by about 30 %. It was therefore decided to use distilled water for the experiments and to skim off the surface film immediately before each measurement. It was usually possible to make a measurement in less than 12 min.

As soon as the surface of the water had been skimmed, the wavemaker was set in motion and the amplitude of its oscillation was determined by measuring the movement of the paddle at a known distance from the hinge. The capacitance transducer was placed at a distance of 20 cm from the wavemaker. When the wavemaker amplitude was large enough to generate a cross-wave the growth of the wave was timed until it reached an amplitude of about 1·5 or 2 mm, and the rate of growth of the wave was determined from these measurements. If, however, no cross-wave was visible after the wavemaker had been vibrating for about 5 min a 'small' disturbance was introduced by gently blowing on the water surface at a distance of about 40 cm from the wavemaker. When a disturbance of this kind died away it was assumed that the amplitude of the wavemaker was below the margin of stability; if the cross-wave built up in amplitude its growth was measured. Then, by interpolating the data on the growth rates as a function of wavemaker amplitude, the point of neutral stability was estimated for the particular frequency of operation.

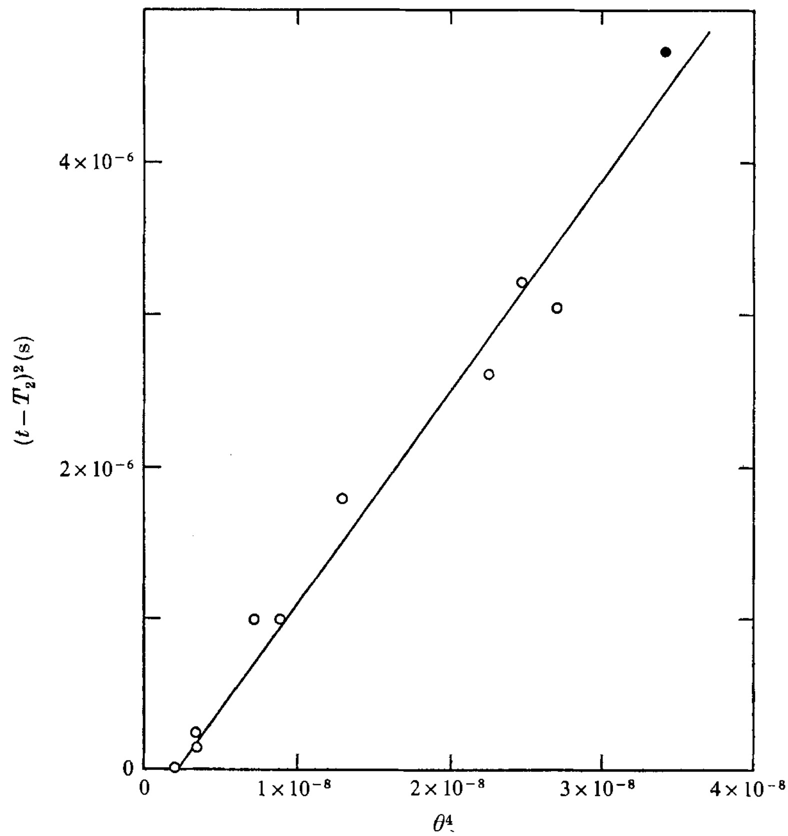


FIGURE 3. The data of figure 2 for the second mode replotted to indicate their conformity with the theoretical result (4). ●, point just below the margin of stability.

The results shown in figure 2 were obtained for the modes $n = 2$ and $n = 3$. The results for the second mode suggest that the set of data is well represented by a symmetric curve. This accordingly yields the period $T_2$ at which the waves were most easily excited. The results for the third mode have more scatter than those for the second, but again the data fit a symmetric curve. The increased scatter associated with these measurements is due to the conditions at the surface of the liquid, which need to be controlled more carefully than in the case of the second mode.

Once estimates have been made of the periods $T_2$ and $T_3$, at which the cross-waves for the respective modes are most easily generated, the results of figure 2 may be compared with the theoretical result given in equation (4). That the results for the second mode agree with the theory can be seen from figure 3, where the graph of the quantity $(\tau - T_2)^2$ as a function of $\theta^4$ conforms very nearly to a linear regression. The slope of this line yields the empirical value of the constant $C_2$, which, according to the theory, is determined by the size of the channel (cf. equation (2)). A further check on the theory can then be made by taking the empirically determined value of $C_2$, together with the values of $T_2$ and $\Theta_2$ obtained from figure 2, and using these to predict a damping rate $L_2$ by

| Mode $n$ | $T_n$ (s) (from figure 1) | $C_n$ (measured) | $C_n$ (equation (2)) | $\Theta_n$ (rad) (from figure 1) | $L_n$ ($s^{-1}$) (equation (3)) | $L_n$ ($s^{-1}$) (measured) |
|---|---|---|---|---|---|---|
| 2 | 0·22202 | 26·9 | 30·8 | 0·0065 | 0·0322 | 0·0311 |
| 3 | 0·18022 | 200 | 147 | 0·0037 | 0·095 | 0·091 |

TABLE 1

means of (3). This may be compared with a direct measurement of $L_2$. The details of these results, and a set for the third mode, are given in table 1.

The agreement between the experimental observations and the theoretical predictions is remarkably good. The shape of the marginal stability curve is well represented by (4), as is indicated in figure 3, and the theoretical value of the constant $C$ agrees well with the measurements. In the case of the second mode the difference is about 14 %, and for the third mode it is 27 % of the measured value of $C$. In the latter case, however, the standard deviation of the least-squares fit of the regression is 23 % of the value of $C$. It should also be noted that the theoretical value of $C$ is quite sensitive to the depth of water in the channel, so that a small error in the determination of $d$ can lead to a relatively large error in the estimate of $C$.

Using the measured values of $\Theta$ to estimate the rate of damping of the cross-wave by means of (3), values of $L$ which are in very good agreement with the direct measurements† were obtained.

### 3.2. *Growth rates*

In the course of determining the margin of stability, the growth rate $\sigma$ was measured, for each of the operating frequencies, at a few values of $\tau$ at which the instability occurs. During the initial stages of development of the cross-waves these rates are in very good agreement with the theory (equation (1)). An example of the agreement obtained with the second mode is shown in figure 4.

As was mentioned in § 1, these theoretical growth rates are expected to apply only during the early development of the cross-waves, whereas at the later stages, under the influence of the second-order pressures, a different mechanism dominates the growth of the waves. In order to demonstrate the extent to which this mechanism increases the growth rates, some measurements were made of

† It is not clear exactly what should be measured in order to determine $L$ directly. If the measurement is made in an open-ended channel, account should be taken of the radiation of the cross-wave field, even though it has a very small group velocity. If, however, the cross-waves are confined within a rectangular box, the measurement is affected by the dissipation at the ends. The measurements quoted in table 1 were made in a closed box. The effect of measuring $L_2$ in this way, rather than in the open-ended channel, was to decrease the observed damping rate of the cross-waves by about 25 %. On the other hand, the dissipation at the false end of the channel is expected, on the basis of a rough calculation of the kind given in Landau & Lifshitz (1959, p. 88), to account for about 7 % of the value of $L_2$ shown in table 1, and about 6 % of $L_3$, for the sizes of the box used in the measurements. The values of $L$ given in table 1 are the mean values of a number of determinations; they are accurate only to within about $\pm 5$ %.

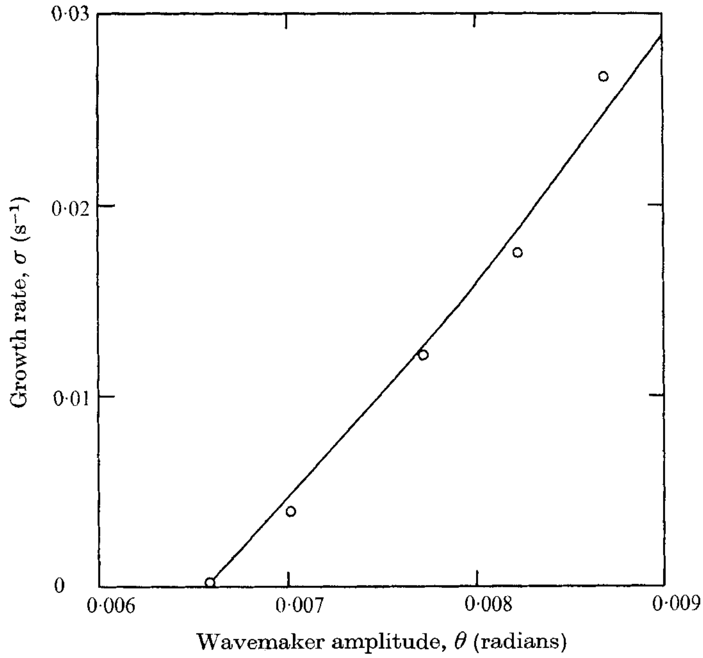


FIGURE 4. The initial growth rate of cross-waves for a fixed period ($\tau = 0{\cdot}22198$ s) within the bandwidth of the second mode. ——, equation (1); ○, experiment.

| Experiment ($n = 2$) | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Wavemaker amplitude, $\theta$ (rad) | 0·0089 | 0·0092 | 0·0099 | 0·0112 |
| Initial growth rate, from equation (1)) $\sigma$ ($s^{-1}$) | 0·028 | 0·032 | 0·042 | 0·062 |
| Measured growth rate ($s^{-1}$) (beyond initial development) | 0·056 | 0·059 | 0·071 | 0·112 |
| Measured decay rate ($s^{-1}$) (during limit cycle) | 0·043 | 0·047 | 0·063 | 0·136 |

TABLE 2. Wavemaker period $\tau = 0{\cdot}22198$ s

the complete development of the waves at the frequency for which the results of figure 4 were obtained. The outcome of these measurements is shown in table 2 and it is seen that the rate of growth of the cross-waves almost doubles as their amplitude builds up.

In these experiments the cross-waves did not reach equilibrium, but passed through cycles of growth and decay as shown in figure 5. It will be seen from table 2 that the measured decay rates during this cyclic process are comparable with the growth rates. This suggests that the decay mechanism may be the same as that responsible for the growth, i.e. an exchange of energy between the wavemaker and the cross-waves. The mechanism based on second-order pressures for the growth of cross-waves depends crucially on the phase relationship between the wavemaker and the established cross-wave, and phase changes could alter the direction in which energy is transferred. It seems probable that energy is

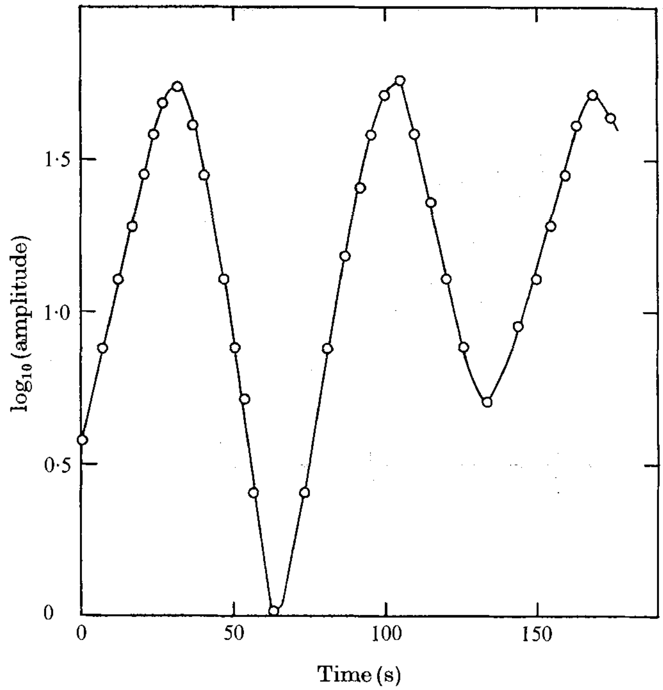


FIGURE 5. The amplitude of a cross-wave ($n = 2$) as a function of time, showing the continual growth and decay of the waves. Wavemaker amplitude $\theta = 0{\cdot}0112$ rad; period $\tau = 0{\cdot}22198$ s. The largest amplitude reached by this wave was 2·75 cm. The zero of time has been chosen arbitrarily.

actively removed from the cross-waves because the decay rates quoted in table 2 exceed the viscous dissipation rate for $n = 2$ (cf. table 1).

Now if the maximum amplitude of the cross-waves had been limited by viscous effects, the phase of the cross-wave relative to a perfect (constant frequency) wavemaker would have been a constant, independent of the amplitude of the wavemaker. The growth and decay process of figure 5 would then have arisen as a consequence of phase drifting of the real wavemaker. But the rate of change of frequency necessary for this to happen in experiment 4 (table 2), which is plotted in figure 5, is approximately 1 part in 500 per minute, persisting for a period of about 35 s. However, the frequency counter indicated maximum drift rates of only about 1 part in 1000 per minute, and mean rates taken over half-minute periods were even smaller.

If, on the other hand, nonlinear effects become important as the amplitude of the cross-waves increases, the growth-and-decay process could be explained as a result of the well-known Poincaré frequency correction for nonlinear standing waves. Thus, as the amplitude of the cross-wave increases, the frequency (and hence the phase) of the cross-wave gradually changes. Such changes would occur at a steady rate, dependent upon the growth rate of the waves, and thus could account for the build up and decay of the cross-waves observed in the experiments.

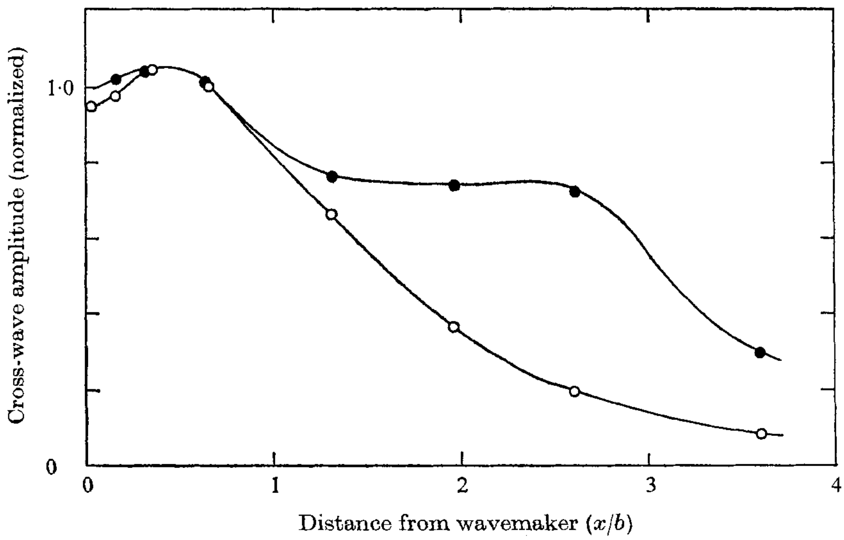


FIGURE 6. Examples of the cross-wave amplitude (for $n = 3$) as a function of the distance from the wavemaker. The amplitudes are given relative to the amplitude of the waves at $x/b = 0{\cdot}65$. Wavemaker period $\tau = 0{\cdot}18070$ s. Wavemaker amplitude $\theta$: ○, 0·0453 rad; ●, 0·0543 rad.

### 3.3. *Distribution along the channel*

Some measurements were made of the variation of the cross-wave amplitude along the channel. In order to make these measurements with the present equipment it was necessary for the cross-waves to reach a state of equilibrium, and although this was not achieved the waves came close to maintaining a steady amplitude when the wavemaker motion was only slightly larger than that at the margin of stability. Operating so that variations in the amplitude of the cross-waves at a given position were less than about 5 %, measurements were made of the cross-wave field along the channel. The results of these measurements, made at two values of the wavemaker amplitude, are given in figure 6; the cross-wave amplitudes are presented in dimensionless form with the amplitude of the cross-wave at a distance of 20 cm from the wavemaker arbitrarily taken to be unity. It is apparent from these (and other) measurements that the amplitude of the cross-wave field decreases more rapidly along the channel at the smaller wavemaker amplitudes. These results are contrary to Mahony's suggestion that the established cross-wave field decays as $\exp(-A\theta x)$, where $x$ is the distance from the wavemaker and $A$ is a constant. The cause of the disagreement may be, however, that the cross-waves never reach a true state of equilibrium since, after the cross-wave amplitude has passed through a maximum, a wave detaches itself from the wavemaker, propagates along the channel and is eventually absorbed at the beach. The process is quite marked at the larger amplitudes. On the other hand, while the cross-wave is building up for the first time, it is clear to the eye that the larger the value of $\theta$, the smaller is the extent of the wave along the channel; this feature is particularly evident with very large values of $\theta$.

### 3.4. *Comments*

It appears that the results obtained from these experiments are in good overall agreement with the theory of Part 1. There are, however, some uncertainties in both the theoretical and the experimental work. The theory is approximate, neglecting second-order terms of the basic non-propagating field in comparison with first-order terms. A fair measure of the relative importance of these terms may be given by $\beta\theta$, where $\beta$ is defined in (2). For the second mode this parameter was no greater than 0·08 and for the third mode it did not exceed 0·12.

The conditions at the surface play an important role in the experiments. To investigate the influence of these conditions, the surface was allowed to stand uncovered overnight and accumulate a fairly thick film. A series of measurements was then conducted in which the amplitude of the wavemaker was set at a fixed level in each case and the frequency was changed in steps. The results were in very good agreement with (1), provided that the frequency $\Omega_2$ was appropriately adjusted, by 0·14 %, suggesting that the dispersion relation was modified slightly by the surface film. Since the change in $\Omega_2$ was fairly small it was felt that an adequate control could be maintained, for the second mode, by cleaning the surface immediately before each measurement.

Other experimental difficulties arise in measuring the damping rate $L$ of the cross-waves. It is not clear precisely how this should be measured, as has already been stated, and so, because of these experimental difficulties, too close an agreement with the theory should not be expected from this measurement. In addition, phase drifting of the wavemaker may cause slower growth rates than those theoretically predicted under the assumption of perfect control over the wavemaker.

We wish to thank Professor T. B. Benjamin for suggesting the experiment and Professor J. J. Mahony for his advice. We are also indebted to Mr J. Bartington for his assistance in developing the apparatus.

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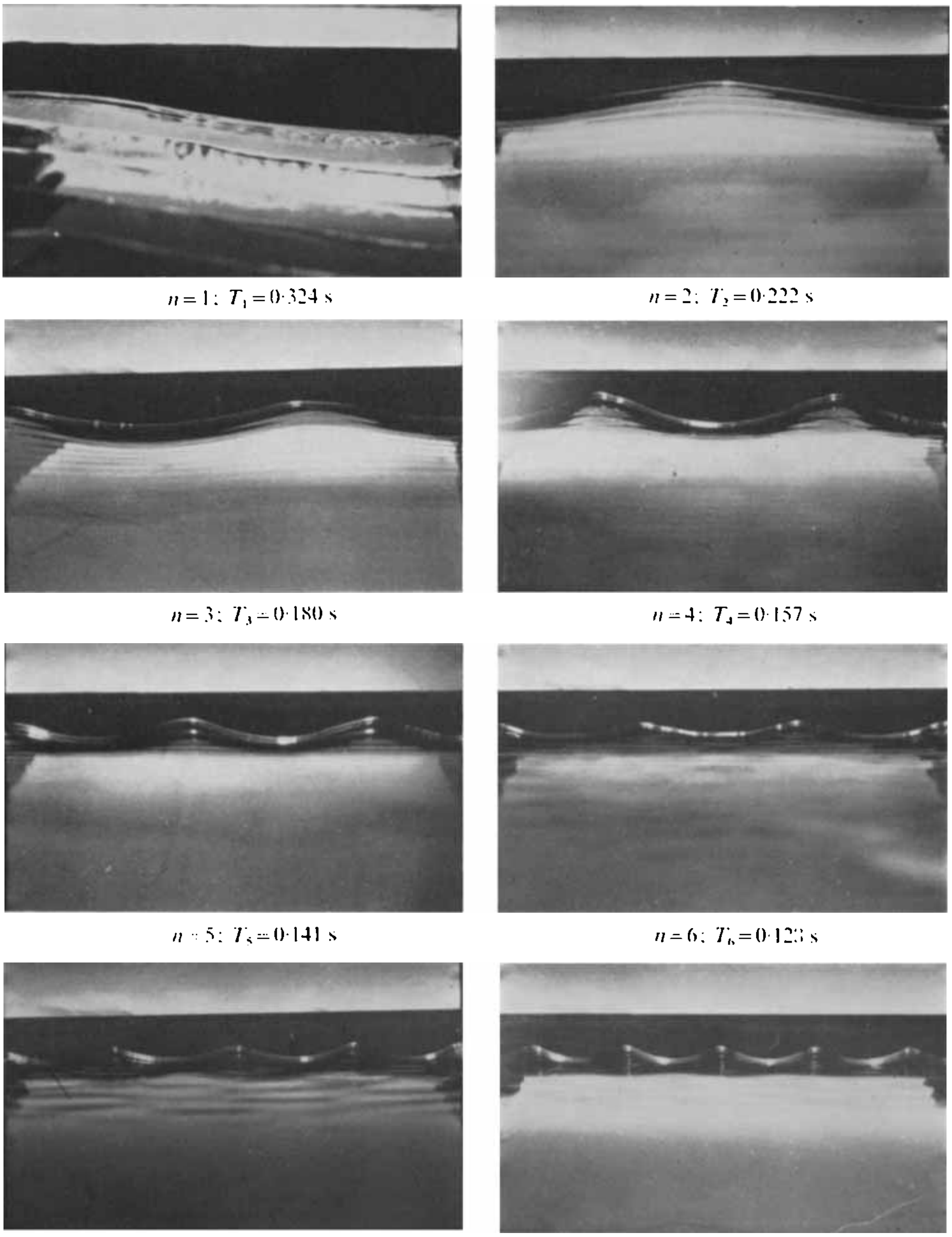

$n = 1$: $T_1 = 0{\cdot}324$ s

$n = 2$: $T_2 = 0{\cdot}222$ s

$n = 3$: $T_3 = 0{\cdot}180$ s

$n = 4$: $T_4 = 0{\cdot}157$ s

$n = 5$: $T_5 = 0{\cdot}141$ s

$n = 6$: $T_6 = 0{\cdot}123$ s

$n = 8$: $T_8 = 0{\cdot}110$ s

$n = 10$: $T_{10} = 0{\cdot}096$ s

FIGURE 1. Photographs of some of the modes of the cross-waves. The view is along the channel towards the wavemaker, which appears as a dark background. The outline of the cross-wave against the wavemaker is clearly visible and its decay along the channel is evident. The sides of the channel are at the edges of the photograph.